UEFA draws: probability calculator

Julien Guyon, Frédéric Meunier
CERMICS, Ecole des Ponts ParisTech

December 13, 2023

In this short note, we explain how the exact draw probability can be computed—and updated live during the actual draw or during a draw simulation. The live probability calculation tool is joint work with Flavien Audibert, Bilel Benaich, Yohann Canavese, and Akram Lamssyah, all second-year students at Ecole des Ponts ParisTech. We also thank Xavier Clerc, Edgar Duc, Antony Feord, and Étienne Polack for their kind contributions.

1 Draw of the round of 16 of the UEFA Champions League (UCL)

The draw has 16 teams: 8 group winners and 8 runners-up. It follows the following rules:

1. group winners are drawn against group runners-up;
2. teams from the same association (country) cannot be drawn against each other;
3. teams from the same group cannot be drawn against each other.

The draw procedure is as follows:

1. A runner-up is randomly drawn.
2. A computer gives the list of admissible opponents for this runner-up.
3. One of the admissible opponents is randomly drawn.
4. The above three steps are repeated until all teams have been drawn.

For Step 2, note that the algorithm considers all possible future draw scenarios in order to prevent that the draw ends in a dead end (a situation where a runner-up has no suitable opponent). For example, if the two runners-up remaining to be drawn are from associations A and B, and the two group winners are from associations B and C, and all teams are from 4 different groups, if the runner-up from association A (RU-A) is drawn, it can a priori be drawn against both group winners. However, the algorithm will select the group winner from association B (W-B)as the unique admissible opponent of RU-A, as if RU-A is drawn against W-C, then RU-B and W-B would play against each other, which is not permitted, as they are both from the same association.

Due to this sequential draw procedure, the draw probabilities are not obvious to compute at all. That is what we do on this website. We explain the maths below. In fact, since the conditional probabilities (once a runner-up has been drawn) are uniform, the (unconditional) draw probabilities are *not* uniform over all possible outcomes of the draw.

2 Draw of the knockout round playoffs of the UEFA Europa League (UEL)

The draw has 16 teams: 8 group runners-up from the UEL and 8 group third-placed teams from the UCL. It follows the following rules:

1. UEL group runners-up are drawn against UCL group third-placed teams;
2. teams from the same association cannot be drawn against each other.

The draw procedure is as follows:

1. A UCL third-placed team is randomly drawn.
2. A computer gives the list of admissible opponents for this team.
3. One of the admissible opponents is randomly drawn.
4. The above three steps are repeated until all teams have been drawn.

3 Draw of the knockout round playoffs of the UEFA Europa Conference League (UECL)

Same as above, with UCL/UEL replaced by UEL/UECL.

4 The mathematics (for the nerds)

The calculation of the exact draw probabilities uses the mathematics of **graph theory**. We take the draw of the round of 16 of the UCL to illustrate the algorithm. We represent the current state of the draw, just before a runner-up is drawn, by a bipartite graph $G = (V, E)$. The set of vertices V represents the teams still to be drawn. E denotes the set of edges: there is an edge between two teams if and only if they are possible opponents (not from the same association and not from the same group). We denote by $L(G)$ the set of the left vertices of G. $L(G)$ is made of the group runners-up still to be drawn. For $v \in V$, we denote by $N(v)$ the set of all neighbors of v in G, i.e., the set of all vertices in G directly connected to v by an edge. $N(v)$ represents the **possible opponents** of team v, i.e., opponents that are not from the same association as v and that were not in the same group as v during the group stage, as per the UEFA rule.

For $i \in L(G)$ we denote by $A(G, i)$ the set of all the neighbors j of i such that the edge ij can be completed into a perfect matching of G; in a perfect matching of G, each runner-up in G (left side of G) is matched with exactly one group winner in G (right side of G). $A(G, i)$ is the set of all **admissible opponents** of the runner-up i, when the set of teams remaining to be drawn is described by G. Indeed, if the edge ij cannot be completed into a perfect matching of G, then j must not be allowed to be drawn against i, since if j is drawn as the opponent of i, the draw will end up in a dead end.

We denote by I a uniformly randomly chosen element in $L(G)$. I represents the randomly chosen runner-up when the set of teams remaining to be drawn is described by G. Following the sequential draw procedure adopted by UEFA, for G a bipartite graph that admits a perfect matching, $i, i_0 \in L(G)$, and $j \in N(i)$, we denote by $\mathbb{P}_G(ij|I = i_0)$ the probability of the edge ij given the graph G and given that the runner-up i_0 was just drawn, and by $\mathbb{P}_G(ij)$ the probability of the edge ij given the graph G. If $|G| > 2$ ($|X|$ denotes the cardinal of the set X), by conditioning on the drawn group winner (paired with i_0) in the case where $i \neq i_0$, we have

$$\mathbb{P}_G(ij|I = i_0) = \begin{cases} \frac{\mathbf{1}_{A(G,i_0)}(j)}{|A(G,i_0)|} & \text{if } i = i_0, \\ \sum_{j_0 \in N(i_0), j_0 \neq j} \frac{\mathbf{1}_{A(G,i_0)}(j_0)}{|A(G,i_0)|} \mathbb{P}_{G \setminus \{i_0, j_0\}}(ij) & \text{if } i \neq i_0. \end{cases} \tag{1}$$

When $|G| = 2$, we must have $i_0 = i$ and $\mathbb{P}_G(ij|I = i_0) = 1$ (since G admits a perfect matching, i and j are connected in G). Moreover, by conditioning on the drawn runner-up I, we have

$$\mathbb{P}_G(ij) = \frac{1}{|L(G)|} \sum_{i_0 \in L(G)} \mathbb{P}_G(ij|I = i_0). \tag{2}$$

All the probabilities $\mathbb{P}_G(ij)$ and $\mathbb{P}_G(ij|I = i_0)$ can thus be computed iteratively for all bipartite subgraphs G of the initial graph G_0, the one that contains the 16 teams (8 group winners and 8 runners-up) that advanced to the round of 16. The bipartite subgraphs correspond to all the possible remaining bipartite subgraphs that can be observed during the draw, as runners-up and group winners are sequentially paired.

Note that in (1), $|A(G, i_0)| > 0$. Indeed, there exists a perfect matching of G_0 (by assumption), and by construction, for any bipartite graph G that admits a perfect matching and for any $j_0 \in A(G, i_0)$, there exists a perfect matching of $G \setminus \{i_0, j_0\}$. As a consequence the iterative formulas (1)-(2) only involve bipartite subgraphs of G_0 that admit perfect matchings.

From (1)-(2), to compute all the probabilities $\mathbb{P}_G(ij)$ and $\mathbb{P}_G(ij|I = i_0)$, it is enough to compute all the sets $A(G, i)$ for G a bipartite subgraph of G_0 that admits a perfect matching and $i \in L(G)$. We compute the $A(G, i)$ by using well-known, available **maximum matching algorithms** from graph theory, rather than recursively. We use the Hopcroft–Karp algorithm.

We avoid as much as possible to do duplicate calculations of the sets of admissible opponents $A(G, i)$ and the probabilities $\mathbb{P}_G(ij)$ and $\mathbb{P}_G(ij|I = i_0)$. If the graph G has already been seen by the algorithm, we just use the stored value of the probabilities. In fact, if we identify that a graph G' that is **isomorphic** to the graph G has already been seen by the algorithm, we just use the stored value of the probabilities $\mathbb{P}_{G'}$, as isomorphic graphs carry the same probabilities (up to a permutation of the vertices, i.e., the teams). With this method, we actually store universal probabilities for this sequential draw procedure of random couplings in bipartite graphs, for small enough graph instances.

Since the probabilities $\mathbb{P}_G(ij)$ and $\mathbb{P}_G(ij|I = i_0)$ are computed for all the subgraphs G that may be encountered during the draw, probabilities can be updated live during the actual draw or during a draw simulation.